

When the set of embeddings is finite?

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Given a manifold N and a number m , we study the following question: *is the set of isotopy classes of embeddings $N \rightarrow S^m$ finite?* In case when the manifold N is a sphere the answer was given by A. Haefliger in 1966. In case when the manifold N is a disjoint union of spheres the answer was given by D. Crowley, S. Ferry and the author in 2011. We consider the next natural case when N is a product of two spheres. In the following theorem, $FCS(i, j) \subset \mathbb{Z}^2$ is a concrete set depending only on the parity of i and j which is defined in the paper.

Theorem. Assume that $m > 2p + q + 2$ and $m < p + 3q/2 + 2$. Then the set of isotopy classes of smooth embeddings $S^p \times S^q \rightarrow S^m$ is infinite if and only if either $q + 1$ or $p + q + 1$ is divisible by 4, or there exists a point (x, y) in the set $FCS(m - p - q, m - q)$ such that $(m - p - q - 2)x + (m - q - 2)y = m - 3$.

Our approach is based on a group structure on the set of embeddings and a new exact sequence, which in some sense reduces the classification of embeddings $S^p \times S^q \rightarrow S^m$ to the classification of embeddings $S^{p+q} \sqcup S^q \rightarrow S^m$ and $D^p \times S^q \rightarrow S^m$. The latter classification problems are reduced to homotopy ones, which are solved rationally.

[57R52](#), [57R40](#); [57R65](#)

1 Introduction

This paper is on the classification of embeddings of higher-dimensional manifolds, see [21] for a recent survey. This generalizes the subject of classical knot theory. In general one can hope only to reduce the isotopy classification problem to problems of homotopy theory [9, 10, 15, 16]. Sometimes the latter can be solved but finding explicit classification is hard.

Given a manifold N and a number m , we study the following simpler question: *is the set of isotopy classes of embeddings $N \rightarrow S^m$ finite?* This question is motivated by analogy to rational homotopy theory founded by J.P. Serre, D. Sullivan and D. Quillen [7] and rational classification of link maps by U. Koschorke [15, 8]. We answer it for simplest manifolds N : spheres, disjoint unions of spheres and products of two spheres. Our main result (Theorem 1.6 below) is an exact sequence, which in some sense reduces the classification of embeddings $S^p \times S^q \rightarrow S^m$ to the classification of embeddings $S^{p+q} \sqcup S^q \rightarrow S^m$ and $D^p \times S^q \rightarrow S^m$. This provides much information about the set of isotopy classes of embeddings $S^p \times S^q \rightarrow S^m$ including a finiteness criterion (Theorem 1.4 below). Throughout the paper we work in smooth category.

This paper concludes the series of papers [2, 3, 4] on this subject. It is independent of previous ones in the sense that it uses statements from [4] but neither definitions nor proofs from any of them.

Knots and links

For *knots* $S^q \rightarrow S^m$ in codimension at least 3 (i.e., $m > q + 2$) the answer to the posed question is known:

Theorem 1.1 [9, Corollary 6.7] *Assume that $m > q + 2$. Then the set of smooth isotopy classes of smooth embeddings $S^q \rightarrow S^m$ is infinite if and only if $m < 3q/2 + 2$ and $q + 1$ is divisible by 4.*

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The classification of (partially) framed knots $D^p \times S^q \rightarrow S^m$ is closely related to the classification of knots.

Theorem 1.2 [4, Corollary 1.14] Assume that $m > q + 2$, $1 \leq p \leq m - q$. Then the set of smooth isotopy classes of smooth embeddings $D^p \times S^q \rightarrow S^m$ is infinite if and only if one of the following conditions holds:

- $4 \mid q + 1$ and $m < p + 3q/2 + 1$;
- $2 \mid q + 1$ and $m = 2q + 1$;
- $2 \mid q$ and $m = p + 2q$.

The classification of links $S^p \sqcup S^q \rightarrow S^m$ is the next natural problem after the classification of knots. For $m \geq 2(p + q)/3 + 2$ there is an explicit description of the isotopy classes of links $S^p \sqcup S^q \rightarrow S^m$ “modulo” knots $S^p \rightarrow S^m$ and $S^q \rightarrow S^m$ in terms of homotopy groups of spheres and Stiefel manifolds; see [9, Theorem 10.7], [24, Theorem 1.1]. In codimension at least 3 there is an exact sequence involving the set of isotopy classes of links and certain homotopy groups [10, Theorem 1.3]. This sequence allows to obtain the following finiteness criterion by D. Crowley, S. Ferry and the author. The criterion involves certain finiteness-checking sets $FCS(i, j) \subset \mathbb{Z}^2$ which depend only on the parity of i, j and which are defined in Table 1 below. A part of each set is drawn in the table; the rest of the set is obtained by obvious periodicity.

Theorem 1.3 [4, Theorem 1.5] Assume that $p, q < m - 2$. Then the set of smooth isotopy classes of smooth embeddings $S^p \sqcup S^q \rightarrow S^m$, whose components are unknotted, is infinite if and only if there exists a point $(x, y) \in FCS(m - p, m - q)$ such that $(m - p - 2)x + (m - q - 2)y = m - 3$.

Table 1: Definition of the finiteness-checking set $FCS(i, j)$ [4, Table 1]

$FCS(i, j)$ is the set of pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $x, y > 0$ and at least one of the following conditions holds —		
for i, j even:	for i odd, j even:	for i, j odd:
<ul style="list-style-type: none"> • $x = 1$ and $y = 1$; • $x = 2$ and $2 \mid y$; • $x = 3$ and $y = 3$; • $x = 3$ and $y \geq 5$; • $x \geq 4$ and $y \geq 4$; • $2 \mid x$ and $y = 2$; • $x \geq 5$ and $y = 3$. 	<ul style="list-style-type: none"> • $x = 1$ and $y = 1$; • $x = 2$ and $2 \mid y + 1$; • $x = 3$ and $y \geq 2$; • $x \geq 4$ and $y \geq 4$; • $4 \mid x$ and $y = 2$; • $4 \mid x + 1$ and $y = 2$; • $x \geq 5$ and $y = 3$. 	<ul style="list-style-type: none"> • $x = 1$ and $y = 1$; • $x = 2$ and $4 \mid y + 2$; • $x = 2$ and $4 \mid y + 3$; • $x \geq 3$ and $y \geq 3$; • $4 \mid x + 2$ and $y = 2$; • $4 \mid x + 3$ and $y = 2$.
For i even, j odd the set $FCS(i, j)$ is obtained from $FCS(j, i)$ by the reflection with respect to the line $x = y$.		

Knotted tori

A natural next step (after link theory and the classification of embeddings of highly-connected manifolds) towards classification of embeddings of arbitrary manifolds is the classification of *knotted tori*, i.e., embeddings $S^p \times S^q \rightarrow S^m$. The classification of knotted tori gives some insight or even precise information concerning arbitrary manifolds [20]; see also Theorem 4.1 below. Many interesting examples of embeddings are knotted tori [13, 17, 19, 14].

There was known an explicit description of the set of knotted tori up to isotopy in the *metastable* dimension $m \geq p + 3q/2 + 2$, $p \leq q$ in terms of homotopy groups of Stiefel manifolds [19, 22]. If N is a closed $(p - 1)$ -connected $(p + q)$ -manifold then until recent results [5, 23, 22] no complete readily calculable descriptions of isotopy classes below the metastable dimension was known, in spite of the existence of interesting approaches of Browder–Wall and Goodwillie–Weiss [25, 6, 1].

The main “practical” result of the paper is an explicit criterion for the finiteness of the set of knotted tori up to isotopy below the metastable dimension:

Theorem 1.4 *Assume that $m > 2p + q + 2$ and $m < p + 3q/2 + 2$. Then the set of isotopy classes of smooth embeddings $S^p \times S^q \rightarrow S^m$ is infinite if and only if at least one of the following conditions holds:*

- $q + 1$ or $p + q + 1$ is divisible by 4,
- there exists a point $(x, y) \in FCS(m - p - q, m - q)$ such that $(m - p - q - 2)x + (m - q - 2)y = m - 3$.

Example 1.5 [3, Example 1] The set of knotted tori $S^1 \times S^5 \rightarrow S^{10}$ is finite.

In Theorem 1.4 the inequality $m < p + 3q/2 + 2$ is assumed by aesthetic reasons — to reduce the number of cases and thus to simplify both the statement and the proof. The classification of knotted tori for $m \geq p + 3q/2 + 2$ is easier and is given by [19, Corollary 1.5], [22, Theorem 1.1], and [4, Lemma 1.12].

A particular case $m > p + 4q/3 + 2$ of Theorem 1.4 was proved in [2, 3] by a different method. To compare roughly the strength of all the mentioned results one could put $p = 1$. Then known results provide a classification for $m > 3q/2 + 3$, while Theorem 1.4 provides a finiteness criterion for $m > q + 4$.

Relationship between framed knots, links and knotted tori

The main result of the paper is an exact sequence (Theorem 1.6 below), which reduces the classification of knotted tori to the classification of links and framed knots.

Let us introduce some notation and conventions. For a smooth manifold N denote by $E^m(N)$ the set of smooth isotopy classes of smooth embeddings $N \rightarrow S^m$. The letter “E” in the notation comes from the word “embedding”. For $m > p + q + 2$ the sets $E^m(S^q)$, $E^m(D^p \times S^q)$, and $E^m(S^{p+q} \sqcup S^q)$ are finitely generated abelian groups with respect to “connected sum”, “framed connected sum”, and “componentwise connected sum” operation, respectively [9, 10]. Denote by $E_0^m(S^{p+q} \sqcup S^q)$ the subgroup of $E^m(S^{p+q} \sqcup S^q)$ formed by all the embeddings $S^{p+q} \sqcup S^q \rightarrow S^m$ whose second component (i.e., restriction to the sphere S^q) is unknotted. For $m > 2p + q + 2$ the “parametric connected sum” operation gives a natural abelian group structure on the set $E^m(S^p \times S^q)$; see Figure 1, Section 2 and paper [22] for details.

Let us state our main “theoretical” result.

Theorem 1.6 *For each $m > 2p + q + 2$ there is an exact sequence of finitely generated abelian groups*

$$\begin{aligned} \dots \rightarrow E_0^m(S^{p+q} \sqcup S^q) \xrightarrow{\sigma^*} E^m(S^p \times S^q) \xrightarrow{i^*} E^m(D^p \times S^q) \xrightarrow{\partial^*} \\ \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1}) \xrightarrow{\sigma^*} E^{m-1}(S^p \times S^{q-1}) \xrightarrow{i^*} E^{m-1}(D^p \times S^{q-1}) \rightarrow \dots \end{aligned}$$

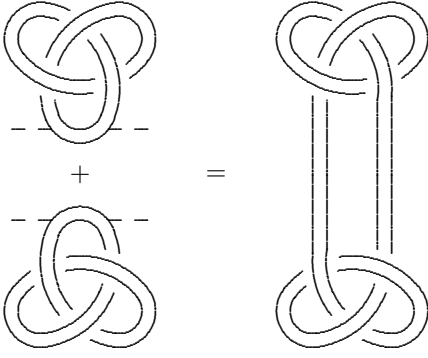


Figure 1: S^1 -parametric connected sum of embeddings $S^1 \times S^1 \rightarrow S^3$ [3, Figure 5]

For $m \geq 3(p+q)/2 + 2$ this sequence is isomorphic to the horizontal sequences in [22, Restriction Lemma 6.3], while for general m our exact sequence can be called a “desuspension” of those ones.

As a corollary, we get the following formula for the rank of the group $E^m(S^p \times S^q)$:

Corollary 1.7 *Assume that $m > 2p + q + 2$ and $m < p + 3q/2 + 2$. Then*

$$\text{rk } E^m(S^p \times S^q) = \text{rk } E^m(S^{p+q} \sqcup S^q) + \text{rk } \pi_q(V_{m-q,p}).$$

Notice that the ranks of the groups in the right-hand side are known [4, Theorem 1.7 and Lemma 1.12].

Organization of the paper

In Section 2 we introduce some notation and required known results. In Section 3 we prove Theorem 1.6. In Section 4 we deduce Theorem 1.4 from Theorem 1.6 and give an easy application of our approach.

The reader who wants to get a nontrivial result in a minimal time may read only subsections “Group Structure”, “Definition of σ^* ”, “Exactness at $E^m(S^p \times S^q)$ ”, and then immediately get the “only if” part of Theorem 1.4 from Theorems 1.1–1.3.

2 Preliminaries

Group structure

Let us begin with the definition of a group structure on the set of knotted tori due to A. Skopenkov [22].

Let $x_1, x_2, \dots, x_m, x_{m+1}$ be the coordinates in the space \mathbb{R}^{m+1} . For each $q \leq m$ identify the space \mathbb{R}^{q+1} with the subspace of \mathbb{R}^{m+1} given by the equations $x_1 = x_2 = \dots = x_{m-q} = 0$. Thus the unit sphere S^q is identified with a subset of S^m . The obtained inclusion is called *the standard embedding* $S^q \rightarrow S^m$. For $p+q+1 \leq m$ *the standard embedding* $s: S^p \times S^q \rightarrow S^m$ is the map given by the formula $((x_1, \dots, x_{p+1}), (y_1, \dots, y_{q+1})) \mapsto (0, \dots, 0, x_1, \dots, x_{p+1}, y_1, \dots, y_{q+1})/\sqrt{2}$, where the number of zero coordinates equals $m - p - q - 1$. It takes the product $S^p \times S^q$ to the boundary of the $\frac{\pi}{4}$ -neighborhood (in the sphere S^{p+q+1}) of the standard embedding $S^q \rightarrow S^{p+q+1}$. The *standard embedding* $s: S^p \times S^q \rightarrow S^m$ is defined by the same formula.

Denote by D_+^q and D_-^q the half-spheres of $S^q \subset \mathbb{R}^{q+1}$ given by the inequalities $x_{q+1} \geq 0$ and $x_{q+1} \leq 0$, respectively. Identify the half-sphere D_-^q with the unit disc D^q by a diffeomorphism $\theta: D^q \rightarrow D_-^q$. Denote by $r_k: \mathbb{R}^m \rightarrow \mathbb{R}^m$ the reflection in the hyperplane given by the equation $x_k = 0$. Denote by $D_{+\epsilon}^q$

(respectively, D_{++}^q) be the subset of the sphere S^q given by the inequality $x_{q+1} \geq \epsilon$ (respectively, by the inequalities $x_{q+1}, x_q \geq 0$). Let $B \subset \text{Int}(D_+^p \times D_+^q)$ be a fixed smooth ball of dimension $p + q$.

If no confusion arise we denote a map and its abbreviation by the same symbol, and also an element and its equivalence class by the same symbol.

Most of the ideas of the paper can be understood from the low-dimensional examples shown in figures. Notice that the proofs may not be literally correct for the shown low dimensions. In the figures instead of the spheres S^1, S^2, S^3 we always show their images under an appropriate stereographic projection.

Definition (See Figure 2 to the left.) A map $f: S^p \times S^q \rightarrow S^m$ is *standardized* if

- $f: S^p \times D_-^q \rightarrow D_-^m$ is the restriction of the standard embedding $s: S^p \times S^q \rightarrow S^m$ and
- $f(S^p \times \text{Int } D_+^q) \subset \text{Int } D_+^m$.

An isotopy $f_t: S^p \times S^q \rightarrow S^m$ is *standardized* if for each $t \in I$ the embedding f_t is standardized.

Lemma 2.1 [22, Standardization Lemma 2.1] Assume that $m > 2p + q + 2$. Then

- (a) any embedding $S^p \times S^q \rightarrow S^m$ is isotopic to a standardized embedding; and
- (b) any isotopy between standardized embeddings $S^p \times S^q \rightarrow S^m$ is isotopic relatively to the ends to a standardized isotopy.

This lemma is equivalent to [22, Lemma 2.1] by the “concordance implies isotopy” theorem [12].

Theorem 2.2 [22, Group Structure Theorem 2.2] Assume that $m > 2p + q + 2$. Then an abelian group structure on the set $E^m(S^p \times S^q)$ is well-defined by the following construction.

- Let $f, g: S^p \times S^q \rightarrow S^m$ be two embeddings. Take standardized embeddings f', g' isotopic to them. By definition, put

$$(f + g)(x, y) = \begin{cases} f'(x, y) & y \in D_+^q \\ r_{m+1}r_m g'(x, r_q r_{q+1} y) & y \in D_-^q. \end{cases}$$

- Set $(-f)(x, y) = r_m f(x, r_q y)$.
- Set 0 to be the standard embedding $s: S^p \times S^q \rightarrow S^m$.

Claim 2.3 [22, Triviality Criterion] Assume that $m > 2p + q + 2$. Then an embedding $S^p \times S^q \rightarrow S^m$ is isotopic to the standard embedding if and only if it extends to an embedding $S^p \times D^{q+1} \rightarrow D^{m+1}$.

The next two subsections give some insight for the proof of Theorem 1.6 although they are not used formally.

Action of knots

Let us define an action of the group of knots on the set of embeddings and prove a particular case of Theorem 1.4. By definition, the map $\sigma^*: E^m(S^{p+q}) \rightarrow E^m(S^p \times S^q)$ takes a knot $f: S^{p+q} \rightarrow S^m$ to the connected sum $f \# s$ of the knot f and the standard embedding $s: S^p \times S^q \rightarrow S^m$. Here we assume that the images of f and s are separated by a hyperplane.

Description of a similar map for arbitrary manifolds is a hard open problem [5]. Fortunately, in our situation enough information can be obtained:

Claim 2.4 [3, Proposition 5.6] For $m > 2p + q + 2$ the map $\sigma^*: E^m(S^{p+q}) \rightarrow E^m(S^p \times S^q)$ is injective.

This claim immediately implies the case “ $p + q + 1$ divisible by 4” of Theorem 1.4, by Theorem 1.1 above. In fact $E^m(S^{p+q})$ is a direct summand in $E^m(S^p \times S^q)$ [22, Smoothing Theorem 2.3].

The proof of Claim 2.4 (and also Theorem 1.6) is based on a surgery over the “torus” $S^p \times S^q$ along a “meridian” $S^p \times y$, $y \in S^q$; see Figure 2. Let us give a formal definition of the *standard surgery*. Fix a diffeomorphism $S^{p+q} \cong S^p \times D_+^q \cup D^{p+1} \times S^{q-1}$. Extend the standard embedding $S^p \times S^{q-1} \rightarrow S^{m-1}$ to an embedding $s': D^{p+1} \times S^{q-1} \rightarrow D_-^m$ meeting the boundary “regularly”, i.e., so that the restriction $s: S^p \times D_+^q \rightarrow D_+^m$ and the embedding $s': D^{p+1} \times S^{q-1} \rightarrow D_-^m$ form together a *smooth map* $S^{p+q} \rightarrow S^m$. Let $f': S^p \times S^q \rightarrow S^m$ be a standardized embedding. Define the result of the *standard surgery* over f' to be the embedding $g: S^{p+q} \rightarrow S^m$ obtained by gluing $f': S^p \times D_+^q \rightarrow D_+^m$ and $s': D^{p+1} \times S^{q-1} \rightarrow D_-^m$ together. Equivalently, $g: S^{p+q} \rightarrow S^m$ is given by the formula

$$g(x) = \begin{cases} f'(x), & x \in S^p \times D_+^q; \\ s'(x), & x \in D^{p+1} \times S^{q-1}. \end{cases}$$

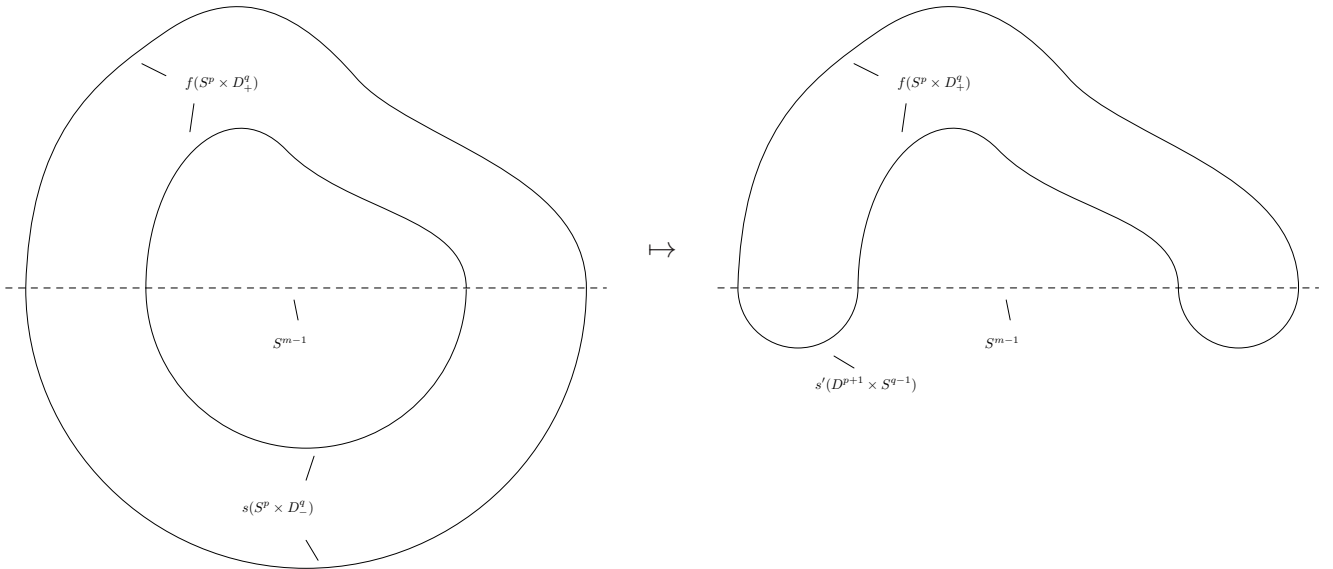


Figure 2: The standard surgery ($p = 0, q = 1, m = 2$)

Proof of Claim 2.4 It suffices to construct a left inverse $\bar{\sigma}^*: E^m(S^p \times S^q) \rightarrow E^m(S^{p+q})$ of the map σ^* . Take an embedding $f: S^p \times S^q \rightarrow S^m$. By Standardization Lemma 2.1.a it is isotopic to a standardized embedding $f': S^p \times S^q \rightarrow S^m$. Set $\bar{\sigma}^*(f)$ be the isotopy class of the embedding $S^{p+q} \rightarrow S^m$ obtained by the standard surgery over f' . The element $\bar{\sigma}^*(f)$ is well-defined by Standardization Lemma 2.1.b. We have $\bar{\sigma}^* \sigma^* = \text{Id}$, because $\bar{\sigma}^* \sigma^*(f) = \bar{\sigma}^*(f \# s) = f \# \bar{\sigma}^*(s) = f \# 0 = f$ for any $f \in E^m(S^{p+q})$. \square

Framed knots

Let us recall an approach to the classification of (partially) framed knots. By a p -framing of an embedded manifold we mean a system of p ordered orthogonal normal unit vector fields on the manifold. Denote by $V_{m-q,p}$ is the *Stiefel manifold* of p -framings of the origin of \mathbb{R}^{m-q} . Clearly, the group $E^m(D^p \times S^q)$ is isomorphic to the group of p -framed embeddings $S^q \rightarrow S^m$ up to p -framed isotopy.

Theorem 2.5 For $m > q + 2$ there is an exact sequence

$$\cdots \rightarrow \pi_q(V_{m-q,p}) \xrightarrow{\tau} E^m(D^p \times S^q) \xrightarrow{i^*} E^m(S^q) \xrightarrow{Ob} \pi_{q-1}(V_{m-q,p}) \rightarrow E^{m-1}(D^p \times S^{q-1}) \rightarrow \cdots$$

The theorem is proved analogously to its particular case $m = p + q$ [9, Corollary 5.9]. We sketch the proof here for convenience of the reader.

Sketch of the proof *Definition of homomorphisms.* The map $i^*: E^m(D^p \times S^q) \rightarrow E^m(S^q)$ is restriction-induced, where $0 \times S^q$ is identified with S^q in an obvious way.

The map $Ob: E^m(S^q) \rightarrow \pi_{q-1}(V_{m-q,p})$ is the complete obstruction to the existence of a p -framing on an embedding $f: S^q \rightarrow S^m$. This obstruction is defined as follows. Take a (unique up to homotopy) $(m - q)$ -framing of the disc fD_+^q . Take a (unique up to homotopy) p -framing of the disc fD_-^q . Thus the sphere fS^{q-1} is equipped both with the p -framing and the $(m - q)$ -framing. Using the $(m - q)$ -framing identify each fiber of the normal bundle to fD_+^q with the space \mathbb{R}^{m-q} . To each point $x \in S^{q-1}$ assign the p -framing at the point fx . This leads to a map $S^{q-1} \rightarrow V_{m-q,p}$. By definition $Ob(f) \in \pi_{q-1}(V_{m-q,p})$ is the homotopy class of this map.

The map $\tau: \pi_q(V_{m-q,p}) \rightarrow E^m(D^p \times S^q)$ is defined as follows. Represent $f \in \pi_q(V_{m-q,p})$ as a smooth map $f: D^p \times S^q \rightarrow D^{m-q}$ linear in each fiber $D^p \times y$, $y \in S^q$. Define $\tau(f)$ to be the composition $D^p \times S^q \rightarrow D^{m-q} \times S^q \rightarrow S^m$ of the embedding $f \times \text{proj}$ and the standard embedding s , i.e., $\tau(f)(x, y) = s(f(x, y), y)$ for each $x \in D^p$, $y \in S^q$.

The exactness at the terms $E^m(D^p \times S^q)$ and $E^m(S^q)$ is checked directly.

Proof of the exactness at the term $\pi_q(V_{m-q,p})$. Let $f: S^{q+1} \rightarrow S^{m+1}$ be an embedding. Then f is isotopic to a *standardized* embedding $f': S^{q+1} \rightarrow S^m$, i.e., satisfying the conditions:

- $f': D_-^{q+1} \rightarrow D_-^{m+1}$ is the restriction of the standard embedding $S^{q+1} \rightarrow S^{m+1}$;
- $f'(\text{Int}D_+^{q+1}) \subset \text{Int}D_+^{m+1}$.

Take a p -framing of the disc $f'(D_+^{q+1})$. Represent this framing by an embedding $g: D^p \times D_+^{q+1} \rightarrow D_+^{m+1}$ linear in each fiber $D^p \times y$, $y \in D_+^{q+1}$. Clearly, $\tau Ob(f') = g|_{D^p \times \partial D_+^{q+1}}$. Thus the embedding $\tau Ob(f'): D^p \times \partial D_+^{q+1} \rightarrow \partial D_+^{m+1}$ extends to the embedding $g: D^p \times D_+^{q+1} \rightarrow D_+^{m+1}$. So $\tau Ob(f')$ is isotopic to the standard embedding $D^p \times S^q \rightarrow S^m$, cf. Claim 2.3 above. Thus $\text{Im } Ob \subset \text{Ker } \tau$. Analogously $\text{Im } Ob \supset \text{Ker } \tau$. \square

3 The exact sequence

First let us define the homomorphisms in Theorem 1.6. These maps are well-defined for $m > p + q + 2$ and are homomorphisms for $m > 2p + q + 2$.

The homomorphism $i^*: E^m(S^p \times S^q) \rightarrow E^m(D^p \times S^q)$ is the composition $E^m(S^p \times S^q) \rightarrow E^m(D_-^p \times S^q) \cong E^m(D^p \times S^q)$ of the restriction-induced map and the map induced by the diffeomorphism $\theta: D^p \cong D_-^p$.

Definition of σ^* .

The map $\sigma^*: E_0^m(S^{p+q} \sqcup S^q) \rightarrow E^m(S^p \times S^q)$ is defined as follows; see Figure 3. Let $f: S^{p+q} \sqcup S^q \rightarrow S^m$ be a smooth embedding, whose restriction to S^q is unknotted. Perform an ambient isotopy making the restriction $f: S^q \rightarrow S^m$ standard. Extend this restriction to the standard embedding $s: D^{p+1} \times S^q \rightarrow S^m$. We may assume that the images fS^{p+q} and $s(D^{p+1} \times S^q)$ are disjoint (because one can always move fS^{p+q} aside a neighbourhood of fS^q). Take a pair of points $x \in fS^{p+q}$ and $y \in s(D_+^{p+1} \times S^q)$. Join them by an arc xy , whose interior misses the images fS^{p+q} and $s(D^{p+1} \times S^q)$. Perform a connected sum of fS^{p+q} and $s(S^p \times S^q)$ along the arc xy . By definition set $\sigma^*(f) \in E^m(S^p \times S^q)$ to be the class of the obtained embedding $S^p \times S^q \rightarrow S^m$.

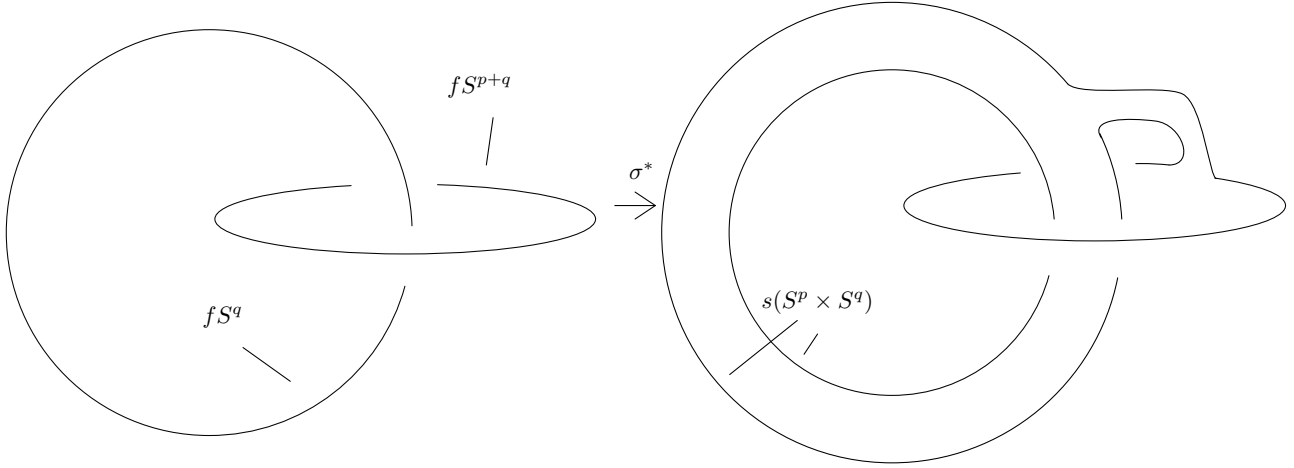


Figure 3: Definition of the map $\sigma^*: E_0^m(S^{p+q} \sqcup S^q) \rightarrow E^m(S^p \times S^q)$ ($p = 0, q = 1, m = 3$)

Claim 3.1 Assume that $m > p + q + 2$. Then the map $\sigma^*: E_0^m(S^{p+q} \sqcup S^q) \rightarrow E^m(S^p \times S^q)$ is well-defined by the above construction.

Proof Let us show that the isotopy class $\sigma^*(f)$ is independent from the choice of the ambient isotopy making the restriction $f: S^q \rightarrow S^m$ standard. It suffices to prove that if two embeddings $f_1, f_2: S^{p+q} \sqcup S^q \rightarrow S^m$ with standard restrictions to S^q are isotopic, then the isotopy can be made fixed on S^q . This is proved in [10, Proof of Theorem 7.1], but let us present the details for the convenience of the reader. Since f_1 and f_2 are isotopic, it follows that there is an orientation-preserving diffeomorphism $h: S^m \rightarrow S^m$ fixed on S^q such that $hf_1 = f_2$. Remove a small neighborhood of a point $y \in S^q$ from S^m ; we get an orientation-preserving embedding $g: D^m \rightarrow S^m$ such that $g(D^m) \supset f_1(S^{p+q})$ and $g(D^m) \cap S^q = g(D^q)$. The embeddings g and hg coincide on D^q . Thus gD^m and hgD^m are tubular neighborhoods of D^q . By uniqueness of tubular neighborhoods and isotopy extension theorem [11] it follows that there is an ambient isotopy $h_t: S^m \rightarrow S^m$ fixed on S^q and such that $h_0 = \text{Id}$, $h_1g = hg$. Then $h_1f_1 = f_2$, hence h_t is the required isotopy. We have proved that the isotopy between f_1 and f_2 can be made fixed on S^q .

For $m > p + q + 2$ the class $\sigma^*(f)$ is independent from the choice of the points x, y and the arc xy . Indeed, take two pairs of points $x_1, x_2 \in fS^{p+q}$ and $y_1, y_2 \in s(D_+^p \times S^q)$ and two arcs x_1y_1, x_2y_2 , whose interior misses the images fS^{p+q} and $s(D^{p+1} \times S^q)$. Clearly, the two arcs can be joined by a general position family of arcs x_ty_t such that $x_t \in fS^{p+q}$ and $y_t \in s(D_+^p \times S^q)$. By general position for $m > p + q + 2$ the interiors of all the arcs x_ty_t miss fS^{p+q} and $s(D^{p+1} \times S^q)$. Let $\sigma_t^*(f): S^p \times S^q \rightarrow S^m$ be the embedding obtained by the connected summation of fS^{p+q} and $s(S^p \times S^q)$ along the arc x_ty_t . Then $\sigma_t^*(f)$ is an isotopy between $\sigma_1^*(f)$ and $\sigma_2^*(f)$. \square

Claim 3.2 Assume that $m > 2p + q + 2$. Then the map $\sigma^*: E_0^m(S^{p+q} \sqcup S^q) \rightarrow E^m(S^p \times S^q)$ is a homomorphism.

Proof Take two embeddings $f_1, f_2 \in E_0^m(S^{p+q} \sqcup S^q)$. Assume without loss of generality that $f_1, f_2|_{S^q}$ are standard embeddings, $f_1(S^{p+q}) \subset D_+^m - s(D^{p+1} \times S^q)$, and $f_2(S^{p+q}) \subset D_-^m - s(D^{p+1} \times S^q)$.

Take three general position arcs x_1y_1, x_2y_2, x_3y_3 such that $x_1, x_2 \in s(D_+^p \times S^q)$, $x_3, y_1 \in f_1(S^{p+q})$, $y_3, y_2 \in f_2(S^{p+q})$. Then the (image of) embedding $\sigma^*(f_1 + f_2)$ is obtained by connected summation of the manifolds $s(S^p \times S^q)$, $f_1(S^{p+q})$, $f_2(S^{p+q})$ along the arcs x_1y_1 and x_3y_3 . The (image of) embedding $\sigma^*(f_1) + \sigma^*(f_2)$ is obtained by connected summation of the manifolds $s(S^p \times S^q)$, $f_1(S^{p+q})$, $f_2(S^{p+q})$ along the arcs x_1y_1 and x_2y_2 . Analogously to the proof of Claim 3.1 it follows that $\sigma^*(f_1 + f_2)$ and $\sigma^*(f_1) + \sigma^*(f_2)$ are isotopic. That is, $\sigma^*(f_1 + f_2) = \sigma^*(f_1) + \sigma^*(f_2)$. \square

Remark The group $E^m(S^{p+q})$ can be identified with the subgroup of the group $E_0^m(S^{p+q} \sqcup S^q)$ formed by all the embeddings with unlinked components. If one makes such an identification then the map $\sigma^*: E_0^m(S^{p+q} \sqcup S^q) \rightarrow E^m(S^p \times S^q)$ extends the map $\sigma^*: E^m(S^{p+q}) \rightarrow E^m(S^p \times S^q)$ defined in §2.

Now we are going to give two equivalent definitions of the map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$. The first definition is easier to understand, while the second one is easier to use in the proof of exactness.

First definition of ∂^*

The map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ is the composition

$$E^m(D^p \times S^q) \xrightarrow{Ob} \pi_{q-1}(S^{m-p-q-1}) \xrightarrow{Ze} E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1}),$$

where the maps Ob and Ze are defined as follows.

The map $Ob: E^m(D^p \times S^q) \rightarrow \pi_{q-1}(S^{m-p-q-1})$ is the complete obstruction to the existence of a unit normal vector field on the embedded manifold $D^p \times S^q$. Let us define the obstruction. Take an embedding $f: D^p \times S^q \rightarrow S^m$. Take a (unique up to homotopy) trivialization of the normal bundle ν to the disc $f(D^p \times D_+^q)$ in S^m . Take a (unique up to homotopy) unit normal vector field on $f(0 \times D_-^q)$. Thus the sphere $f(0 \times S^{q-1})$ is equipped both with a normal vector field and with a trivialization of the restriction of the normal bundle ν . Using the trivialization identify each fiber of the normal bundle ν with the space \mathbb{R}^{m-p-q} . To each point $x \in S^{q-1}$ assign the unit vector of the field at the point $f(0 \times x)$. Thus a map $S^{q-1} \rightarrow S^{m-p-q-1}$ is defined. By definition, $Ob(f) \in \pi_{q-1}(S^{m-p-q-1})$ is the homotopy class of this map.

The map $Ze: \pi_{q-1}(S^{m-p-q-1}) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ is the *Zeeman construction* of the link with a given linking number. To define the map, consider the triple of nested spheres $S^{q-1} \subset S^{p+q-1} \subset S^{m-1}$. Take a trivialization of the normal bundle to the disc D_+^{p+q-1} in S^{m-1} . Analogously to the above each element $f \in \pi_{q-1}(S^{m-p-q-1})$ determines (up to homotopy) a unit vector field on S^{q-1} normal to S^{p+q-1} . Push the sphere S^{q-1} in the direction of this vector field. Let $Ze(f)$ be the link formed by the obtained embedding $S^{q-1} \rightarrow S^{m-1}$ and the standard embedding $S^{p+q-1} \rightarrow S^{m-1}$.

Claim 3.3 Assume $m > p + q + 2$. Then the map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ is a homomorphism.

Proof It suffices to show that both maps Ob and Ze are homomorphisms.

To prove that Ob is a homomorphism, take two embeddings $f_1, f_2: D^p \times S^q \rightarrow S^m$. Clearly, they are isotopic to embeddings f'_1, f'_2 satisfying the following properties for each $i = 1, 2$:

- $f'_i: D^p \times (S^q - D_{++}^q) \rightarrow S^m - D_{++}^m$ is the restriction of the standard embedding $s: D^p \times S^q \rightarrow S^m$;
- $f'_i(D^p \times \text{Int}D_{++}^q) \subset \text{Int}D_{++}^m$.

The sum $f_1 + f_2$ is isotopic to the embedding $f: D^p \times S^q \rightarrow S^m$ given by the formula:

$$f(x, y) = \begin{cases} f'_1(x, y) & y \in D_+^q \\ r_m r_{m+1} f'_2(x, r_q r_{q+1} y) & y \in D_-^q. \end{cases}$$

For this particular construction of the sum the formula $Ob(f_1 + f_2) = Ob(f_1) + Ob(f_2)$ is checked directly.

It is also easy to prove that Ze is a homomorphism, cf. [10, Theorem 10.1]. \square

Second definition of ∂^* .

The map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ will be an obstruction to extend an embedding $D^p \times S^q \rightarrow S^m$ to an embedding $S^p \times S^q \rightarrow S^m$. As usual we identify $D^p \times S^q$ with $D_-^p \times S^q$ by the diffeomorphism $(x, y) \mapsto (\theta x, y)$. To define this obstruction we need a claim, an auxillary definition and a lemma. Recall that $B \subset D_+^p \times D_+^q$ is a fixed ball.

Claim 3.4 Assume that $m > 2p + q + 2$. Then

- (a) any embedding $D_-^p \times S^q \rightarrow S^m$ extends to an embedding $S^p \times S^q - \text{Int } B \rightarrow S^m$;
- (b) any two embeddings $S^p \times S^q - \text{Int } B \rightarrow S^m$, whose restrictions to $D_-^p \times S^q$ are isotopic, are also isotopic.

Proof (a) Let $f: D_-^p \times S^q \rightarrow S^m$ be an embedding. Take a point $y \in D_-^q$. Extend the embedding $f: \partial D_-^p \times y \rightarrow S^m$ to a general position map $g: D_+^p \times y \rightarrow S^m$ meeting the boundary regularly, i.e., such that the maps $f: D_-^p \times y \rightarrow S^m$ and $g: D_+^p \times y \rightarrow S^m$ form together a smooth map $S^p \times y \rightarrow S^m$. Since $m > 2p + q + 2$ it follows that the map g does not have self-intersections and does not intersect the image $f(D_-^p \times S^q)$. Gluing together the embeddings $f: D_-^p \times S^q \rightarrow S^m$ and $g: D_+^p \times y \rightarrow S^m$ we get an embedding $D_-^p \times S^q \cup D_+^p \times y \rightarrow S^m$.

The restriction $f: \partial D_-^p \times D_-^q \rightarrow S^m$ defines a q -framing of the sphere $f(\partial D_-^p \times y)$. The complete obstruction to extension of this q -framing to a q -framing of the disc $g(D_+^p \times y)$ belongs to the group $\pi_{p-1}(V_{m-p,q})$. Since $m > 2p + q + 2$ it follows that the latter group vanishes. Thus the q -framing of the sphere $f(\partial D_-^p \times y)$ extends to a q -framing of the disc $g(D_+^p \times y)$. The latter q -framing defines an embedding $D_-^p \times S^q \cup D_+^p \times D_-^q \rightarrow S^m$ extending the embedding $f: D_-^p \times S^q \rightarrow S^m$.

Clearly, the obtained embedding $D_-^p \times S^q \cup D_+^p \times D_-^q \rightarrow S^m$ extends to an embedding $S^p \times S^q - \text{Int } B \rightarrow S^m$.

(b) This is proved by a relative version of the argument from point (a). \square

Definition (See Figure 4 to the top.) An embedding $F: S^p \times S^q - \text{Int } B \rightarrow S^m$ is *B-standardized* if

- (1) $F: S^p \times D_-^q \rightarrow D_-^m$ is the restriction of the standard embedding $s: S^p \times S^q \rightarrow S^m$;
- (2) $F(S^p \times \text{Int } D_+^q - B) \subset \text{Int } D_+^m$;
- (3) $F(\partial B) \subset \partial D_-^m$; and
- (4) $F(\partial B) \cap s(D^{p+1} \times S^{q-1}) = \emptyset$.

An isotopy $F_t: S^p \times S^q - \text{Int } B \rightarrow S^m$ is *B-standardized* if for each $t \in I$ the embedding F_t is *B-standardized*. A *B-standardized* embedding $F: S^p \times S^q \rightarrow S^m$ is defined analogously, only the above properties (3) and (4) are replaced by

- (3') $F(\text{Int } B) \subset \text{Int } D_-^m$;
- (4') $F(\text{Int } B) \cap s(D^{p+1} \times D_-^q) = \emptyset$.

Lemma 3.5 Assume that $m > 2p + q + 2$. Then

- (a) any embedding $S^p \times S^q - \text{Int } B \rightarrow S^m$ is isotopic to a *B-standardized* embedding;
- (b) any embedding $S^p \times S^q \rightarrow S^m$ is isotopic to a *B-standardized* embedding; and
- (c) any isotopy between *B-standardized* embeddings $S^p \times S^q - \text{Int } B \rightarrow S^m$ is isotopic relative to the ends to a *B-standardized* isotopy.

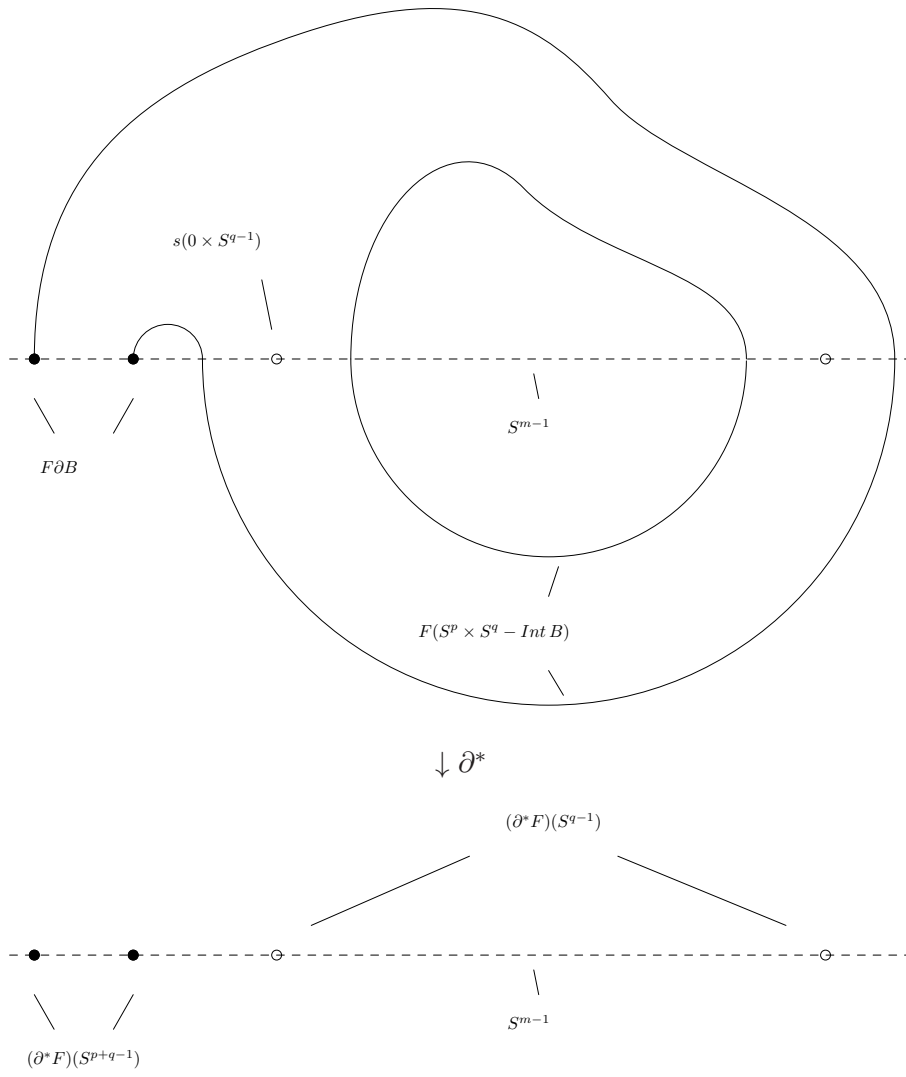
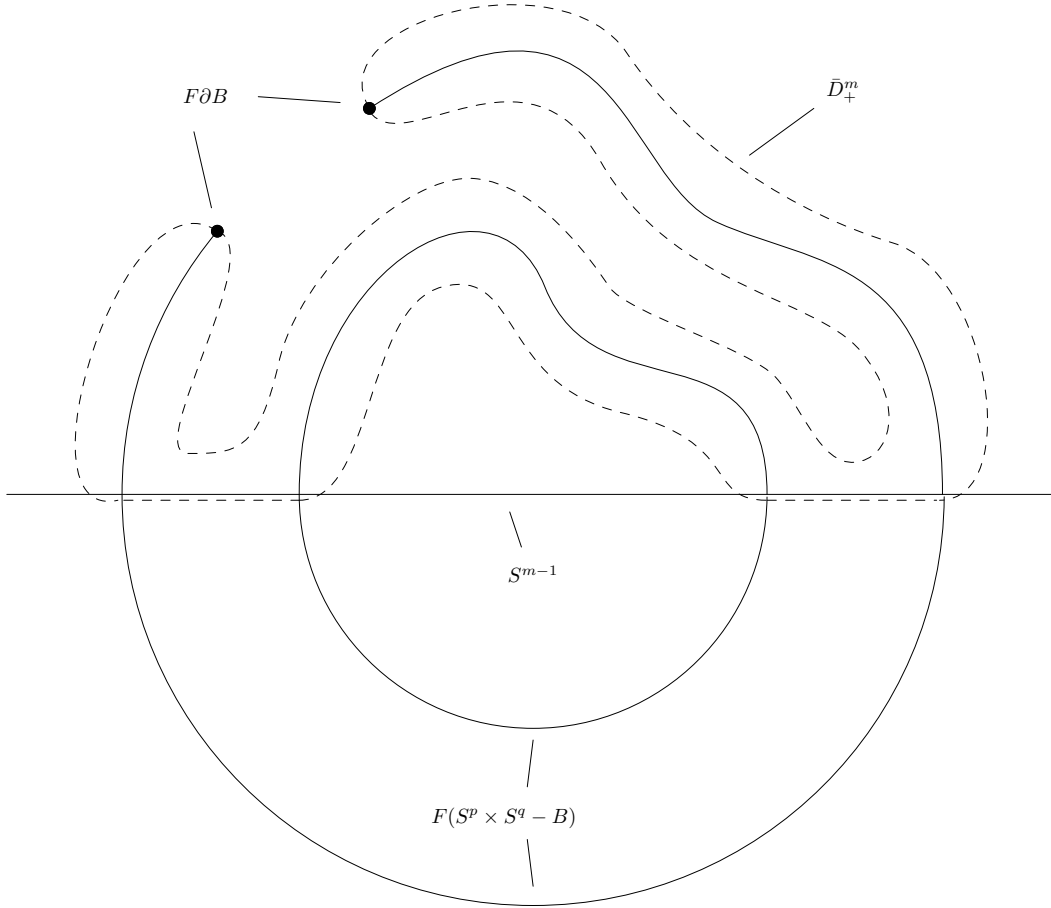


Figure 4: The second definition of the map ∂^* ($p=0, q=1, m=2$)

Figure 5: Making an embedding $(S^0 \times S^1 - B) \rightarrow S^2$ B -standardized

Proof (a) Take an embedding $F: S^p \times S^q - \text{Int } B \rightarrow S^m$. By a generalization of Lemma 2.1 (Lemma 4.2 below) F is isotopic to an embedding $F': S^p \times S^q - \text{Int } B \rightarrow S^m$ satisfying properties (1) and (2) of a B -standardized embedding.

In order to achieve properties (3) and (4) we argue by the following plan. First we construct a neighborhood $\bar{D}_+^m \subset D_+^m$ of (certain smoothing of) the disc $F'(S^p \times D_+^q - \text{Int } B) \cup s(D^{p+1} \times S^{q-1})$, see Figure 5. Then we perform an ambient isotopy $H_t: S^m \rightarrow S^m$, where $t \in [0, 1]$, fixed on $s(D^{p+1} \times D_-^q)$ and taking the neighborhood \bar{D}_+^m to D_+^m . This isotopy takes Figure 5 to the top part of Figure 4. The obtained embedding $H_1 F': S^p \times S^q - \text{Int } B \rightarrow S^m$ will be B -standardized.

To construct the neighborhood \bar{D}_+^m perform the following version of the standard embedded surgery over F' (slightly different from the version defined in § 2). Take a number $0 < \epsilon < 1$. Fix a diffeomorphism $S^{p+q} \cong S^p \times D_{+\epsilon}^q \cup D^{p+1} \times S^{q-1}$. Assume without loss of generality that $F' = s$ on $S^p \times (S^q - D_{+\epsilon}^q)$. Take a smooth embedding $s': D^{p+1} \times S^{q-1} \rightarrow \text{Int } D_+^m$ close to $s: D^{p+1} \times S^{q-1} \rightarrow S^{m-1}$ “meeting the boundary regularly” (i.e., such that $s: S^p \times D_{+\epsilon}^q \rightarrow D_+^m$ and $s': D^{p+1} \times S^{q-1} \rightarrow D_-^m$ form together a smooth map $S^{p+q} \rightarrow S^m$). Gluing together the embedding $s': D^{p+1} \times S^{q-1} \rightarrow \text{Int } D_+^m$ and the restriction $F': S^p \times D_{+\epsilon}^q - \text{Int } B \rightarrow D_+^m$, we get an embedding $G: D^{p+q} \rightarrow S^m$. By definition, this is the result of the version of *the standard surgery*.

Take a framing of the disc GD^{p+q} . Take a tubular neighborhood \bar{D}_+^m of GD^{p+q} in S^m defined by this framing. Then $F'B \subset \partial \bar{D}_+^m$. Assume also without loss of generality that $s(D^{p+1} \times S^{q-1}) \subset \partial \bar{D}_+^m$.

Let us construct an ambient isotopy $H_t: S^m \rightarrow S^m$ such that $H_1 \bar{D}_+^m = D_+^m$ and $H_1|_{s(D^{p+1} \times S^q)} = \text{Id}$. This construction is analogous to the proof of Claim 3.1 above. Both \bar{D}_+^m and D_+^m are tubular neighborhoods of

$F(x \times D_+^q)$, where $x \in D_-^p$. By uniqueness of tubular neighborhoods and isotopy extension theorem [11] there is an ambient isotopy $H_t: S^m \rightarrow S^m$ fixed on $s(S^p \times D_-^q)$ and such that $H_1 \bar{D}_+^m = D_+^m$.

Thus the embedding $H_1 F': S^p \times S^q - \text{Int } B \rightarrow S^m$ satisfies all the above properties (1)–(4). Assertion (a) is proved.

(b), (c) are proved analogously. \square

Now we are ready to give the second definition of the map ∂^* . Take an embedding $f: D^p \times S^q \rightarrow S^m$. Extend it to an embedding $F: S^p \times S^q - \text{Int } B \rightarrow S^m$. Take a B -standardized embedding $F': S^p \times S^q - \text{Int } B \rightarrow S^m$ isotopic to F . Identify ∂B with S^{p+q-1} . Set $\partial^* f$ to be the disjoint union of the restriction $F': \partial B \rightarrow S^{m-1}$ and the standard embedding $S^{q-1} \rightarrow S^{m-1}$.

Claim 3.6 Assume $m > 2p+q+2$. Then the map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ is well-defined by the above construction.

Proof The construction is possible by Claim 3.4.a and Lemma 3.5.a. The result of the construction does not depend on the choice of the extension $F: S^p \times S^q - \text{Int } B \rightarrow S^m$ of the given embedding $f: D^p \times S^q \rightarrow S^m$ by Claim 3.4.b. The result does not depend on the choice of the B -standardization $F': S^p \times S^q - \text{Int } B \rightarrow S^m$ by Lemma 3.5.c. The result depends only on the isotopy class of the embedding $f: D^p \times S^q \rightarrow S^m$ by Claim 3.4.b and Lemma 3.5.c. \square

Claim 3.7 The two given definitions of the map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ are equivalent.

In fact neither the first definition of ∂^* nor this claim are used in the proof of Theorem 1.6.

Proof of Claim 3.7 For a while denote by ∂_I^* and ∂_{II}^* the maps given by the first and the second definitions of ∂^* , respectively. Let $f: D^p \times S^q \rightarrow S^m$ be an embedding. By Claim 3.4 it extends to an embedding $F: S^p \times S^q - \text{Int } B \rightarrow S^m$. By a generalization of Lemma 2.1 the embedding F is isotopic to an embedding $F': S^p \times S^q - \text{Int } B \rightarrow S^m$ satisfying properties (1) and (2) of a B -standardized embedding.

Recall some notation from the proof of Lemma 3.5. Let $s': D^{p+1} \times S^{q-1} \rightarrow \text{Int } D_+^m$ be a smooth embedding close to $s: D^{p+1} \times S^{q-1} \rightarrow S^{m-1}$ such that gluing together $s': D^{p+1} \times S^{q-1} \rightarrow \text{Int } D_+^m$ and $F': S^p \times D_{+\epsilon}^q - \text{Int } B \rightarrow D_+^m$ one gets a smooth embedding $G: D^{p+q} \rightarrow S^m$. Let \bar{D}_+^m be a tubular neighborhood of GD^{p+q} .

Let ν be the unit vector field on $s'(0 \times S^{q-1})$ orthogonal to $\mathbb{R}^m \subset \mathbb{R}^{m+1}$ and looking towards D_-^m . Clearly, ν is orthogonal to GD^{p+q} . A framing of the disc GD^{p+q} identifies the fibers of the normal bundle to the disc with \mathbb{R}^{m-p-q} . Thus the vector field ν determines an element $a \in \pi_{q-1}(S^{m-p-q-1})$.

Let us show that $a = \text{Ob}(f)$. Indeed, extend the vector field ν to the unit normal field on $s'(D^{p+1} \times S^{q-1})$ parallel to the subspace $\mathbb{R}^{p+q+1} \subset \mathbb{R}^{m+1}$. The sphere $s'(y \times S^{q-1})$, where y is the north pole of the sphere S^p , is equipped both with the extended vector field ν and the above framing of the disc GD^{p+q} . This equipment determines an element $b \in \pi_{q-1}(S^{m-p-q-1})$. On one hand, clearly $b = a$. On the other hand, the restriction of the vector field ν to the sphere $s'(y \times S^{q-1}) = s(y \times \partial D_{+\epsilon}^q)$ extends to the disc $s(y \times (D^q - \text{Int } D_{+\epsilon}^q))$ and the above framing is well-defined on a subdisc $f(D^p \times D_{+\epsilon}^q) \subset GD^{p+q}$. Hence $b = \text{Ob}(f)$ by definition. Thus $a = \text{Ob}(f)$.

Push the sphere $s'(0 \times S^{q-1})$ towards the vector field ν until it lies in $\partial \bar{D}_+^m$. Let $g: S^{q-1} \rightarrow \partial \bar{D}_+^m$ be the obtained embedding. Clearly, $\partial_{II}^* f$ is the isotopy class of the link $H_1 F \sqcup H_1 g: \partial B \sqcup S^{q-1} \rightarrow S^{m-1}$.

Take an ambient isotopy $h_t: S^m \rightarrow S^m$ satisfying the following properties:

- $h_1 \partial \bar{D}_+^m = S^{m-1}$;

- $h_1(GD^{p+q}) = D^{p+q}$;
- h_1 takes the fibers of the normal bundle to GD^{p+q} to discs orthogonal to D^{p+q} .

Then the link $h_1F \sqcup h_1g: \partial B \sqcup S^{q-1} \rightarrow S^{m-1}$ is isotopic to $\partial_{lf}^* f$.

Let us show that the link $h_1F \sqcup h_1g: \partial B \sqcup S^{q-1} \rightarrow S^{m-1}$ is also isotopic to the link $\text{ZeOb}(f)$. Indeed, the embedding $s': 0 \times S^{q-1} \rightarrow GD^{p+q}$ is unknotted because it extends to an embedding $0 \times D^q \rightarrow GD^{p+q}$. Thus without loss of generality one can assume that $h_1s'(0 \times S^{q-1})$ is a homothety of S^{q-1} , more precisely, $h_1s'(0 \times x) = x/2$ for each $x \in S^{q-1}$. Equip the sphere S^{q-1} with the unit normal vector field u given by the formula $u(x) = h_1^*v(s'(y \times x))$ for each $x \in S^{q-1}$. Then the result of pushing the sphere S^{q-1} along the vector field $u(x)$ is the same as of pushing the sphere $h_1s'(0 \times S^{q-1})$ along the vector field h_1^*v . Thus $\partial_{lf}^* = \text{ZeOb} = \partial_f^*$. \square

Exactness at $E^m(S^p \times S^q)$

Proof that $\text{Im } \sigma^* \subset \text{Ker } i^*$ Let $f: S^{p+q} \sqcup S^q \rightarrow S^m$ be an embedding, whose restriction to S^q is unknotted. Then the restriction of $\sigma^*(f)$ to $D_-^p \times S^q$ coincides with the restriction of the standard embedding $s: S^p \times S^q \rightarrow S^m$. Thus $i^*\sigma^*(f) = 0$. \square

Proof that $\text{Im } \sigma^* \supset \text{Ker } i^*$ Let $f: S^p \times S^q \rightarrow S^m$ be an embedding whose restriction to $D_-^p \times S^q$ is isotopic to the standard embedding. Let us construct an embedding $g \in E_0^m(S^{p+q} \sqcup S^q)$ such that $\sigma^*g = f$.

Let us give the plan of the proof. First we perform an isotopy making the restriction of the embedding f to $S^p \times S^q - \text{Int } B$ standard. A possible result is shown in the first “frame” of Figure 6. Then we remove the intersection of the image fB with $s(\text{Int } D^{p+1} \times S^q)$ by an isotopy. The result is shown in the second “frame” of Figure 6. A surgery of the obtained embedding gives the required link $g: S^{p+q} \sqcup S^q \rightarrow S^m$. The process of the surgery is shown in the third “frame” of Figure 6. Clearly, the embedding in Figure 6 is of the form σ^*g for some embedding $g: S^{p+q} \sqcup S^q \rightarrow S^m$.

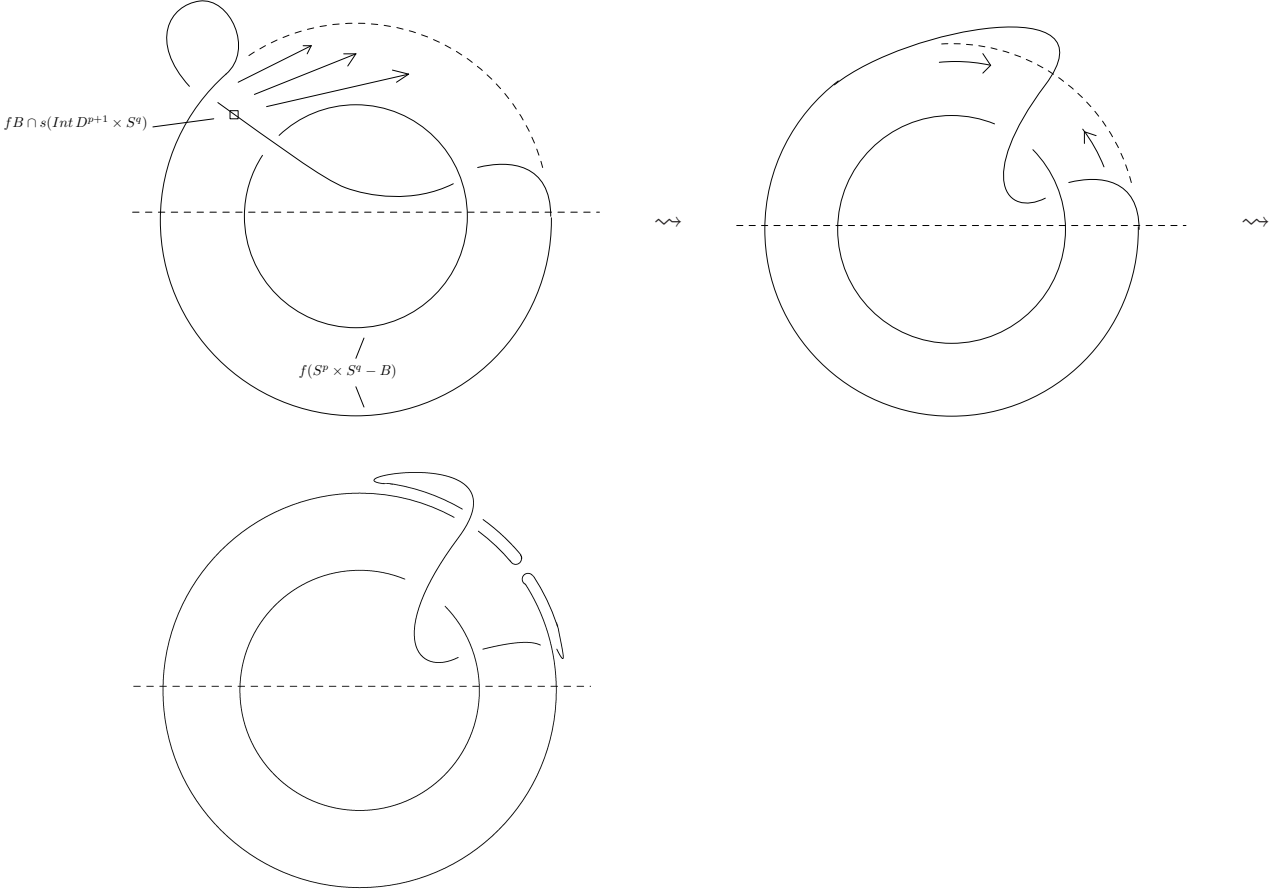
Let us make the restriction $f: S^p \times S^q - \text{Int } B \rightarrow S^m$ standard. By Standardization Lemma 2.1.a one can make the embedding $f: S^p \times S^q \rightarrow S^m$ standardized. Thus f and s will coincide on $S^p \times D_+^p$. Since the restriction of f to $D_-^p \times S^q$ is isotopic to the standard embedding, by isotopy extension theorem it follows that there is an isotopy of f fixed on $S^p \times D_+^p$ making f standard on $D_-^p \times S^q$. Again by isotopy extension theorem one can make f and s equal also on $S^p \times S^q - \text{Int } B$. Denote the embedding obtained after all the above isotopies still by $f: S^p \times S^q \rightarrow S^m$.

Let us remove the intersection of fB with the image $s(\text{Int } D^{p+1} \times S^q)$. Since f is standardized it follows that this intersection is a subset of the ball $s(\text{Int } D^{p+1} \times \text{Int } D_+^q)$. Clearly there exist an ambient isotopy fixed on $f(S^p \times D_+^p \cup D_-^p \times S^q)$ moving this intersection toward the face $s(B)$ of the ball $s(D^{p+1} \times D_+^q)$ and after all removing it. Denote the embedding $S^p \times S^q \rightarrow S^m$ obtained by this isotopy still by f .

Let us perform a surgery over f . Fix a diffeomorphism $S^{p+q} \cong B \sqcup D_-^{p+q}$. Take an embedding $s'': D_-^{p+q} \rightarrow s((D^{p+1} - 0) \times S^q) \subset S^m$ meeting the boundary “regularly”, i.e., so that $s: B \subset S^p \times S^q \rightarrow S^m$ and $s'': D_-^{p+q} \rightarrow S^m$ form together a *smooth unknotted* embedding $S^{p+q} \rightarrow S^m$. Define an embedding $g: S^{p+q} \rightarrow S^m$ to be the result of gluing $s'': D_-^{p+q} \rightarrow S^m$ and $f: B \rightarrow S^m$ together. Formally, put

$$g(x) = \begin{cases} fx & \text{for } x \in B; \\ s''x & \text{for } x \in D_-^{p+q}. \end{cases}$$

Extend the embedding $g: S^{p+q} \rightarrow S^m$ to the link $g: S^{p+q} \sqcup S^q \rightarrow S^m$, whose restriction to the second component S^q is standard. Clearly, the embeddings $\sigma^*g: S^p \times S^q \rightarrow S^m$ and $f: S^p \times S^q \rightarrow S^m$ are joined by an ambient isotopy fixed outside a small neighborhood of the disc $s(B)$. Thus $\sigma^*g = f$. \square

Figure 6: A movie-proof that $\text{Im } \sigma^* \supset \text{Ker } i^*$ ($p = 0, q = 1, m = 3$)**Exactness at $E^m(D^p \times S^q)$**

Proof that $\text{Im } i^* \subset \text{Ker } \partial^*$ Take an embedding $f: S^p \times S^q \rightarrow S^m$. By Lemma 3.5.b it follows that f is isotopic to a B -standardized embedding $f': S^p \times S^q \rightarrow S^m$. By properties (3') and (4') of a B -standardized map it follows that the embedding $\partial^* i^* f = f' \sqcup s: \partial B \sqcup 0 \times S^{q-1} \rightarrow S^{m-1}$ extends to the embedding $f': B \sqcup 0 \times D_-^q \rightarrow D_-^m$. Thus the link $\partial^* i^* f$ is trivial. Hence $\partial^* i^* = 0$. \square

Proof that $\text{Im } i^* \supset \text{Ker } \partial^*$ Let $f: D^p \times S^q \rightarrow S^m$ be an embedding such that $\partial^* f = 0$. By Claim 3.4.a it extends to an embedding $F: S^p \times S^q - \text{Int } B \rightarrow S^m$. By Lemma 3.5.a the embedding F is isotopic to a B -standardized embedding $F': S^p \times S^q - \text{Int } B \rightarrow S^m$. Since $\partial^* f = 0$ it follows that the link $F' \sqcup s: \partial B \sqcup (0 \times S^{q-1})$ is trivial. Hence the embedding $F': \partial B \rightarrow \partial D_-^m - s(D^{p+1} \times S^{q-1})$ extends to an embedding $G: B \rightarrow D_-^m - s(D^{p+1} \times D_-^q)$. Without loss of generality we may assume that it meets the boundary “regularly”, i.e., the embeddings $G: B \rightarrow S^m$ and $F': S^p \times S^q - \text{Int } B \rightarrow S^m$ form together a smooth embedding $S^p \times S^q \rightarrow S^m$. Gluing together the embeddings F' and G we get an embedding $G': S^p \times S^q \rightarrow S^m$. Clearly, $i^* G' = f$. \square

Exactness at $E_0^m(S^p \sqcup S^q)$

Proof that $\text{Im } \partial^* \subset \text{Ker } \sigma^*$ Let $f: D^p \times S^{q+1} \rightarrow S^{m+1}$ be an embedding. By Claim 3.4.a it extends to an embedding $F: S^p \times S^{q+1} - \text{Int } B \rightarrow S^{m+1}$. By Lemma 3.5.a the embedding F is isotopic to a B -standardized embedding $F': S^p \times S^{q+1} - \text{Int } B \rightarrow S^{m+1}$. Take a pair of points $x \in F' \partial B$ and $y \in F'(D_+^p \times S^q)$. Join them

by two generic arcs: $l \subset \partial D_+^m$ and $l' \subset \text{Im } F'$. Span the union $l \cup l'$ by a generic 2-disc $L \subset D_+^{m+1}$ with corners at x and y . Take an arbitrary framing of L . Perform an embedded surgery over F' along the framed disc L . We get an embedding $G: S^p \times D^{q+1} \rightarrow D_+^{m+1}$ whose restriction to the boundary is the connected sum of $F': \partial B \rightarrow S^m$ and $F': S^p \times S^q \rightarrow S^m$ along the arc l . Clearly, the latter connected sum is $\sigma^* \partial^* f$. By Claim 2.3 it follows that the connected sum is isotopic to the standard embedding $S^p \times S^q \rightarrow S^m$. Thus $\sigma^* \partial^* = 0$. \square

Proof that $\text{Im } \partial^* \supset \text{Ker } \sigma^*$ Take an embedding $f: S^{p+q} \sqcup S^q \rightarrow S^m$ whose restriction to S^q is unknotted and such that $\sigma^*(f) = 0$. Assume without loss of generality that $f: S^q \rightarrow S^m$ is standard and $fS^{p+q} \cap s(D^{p+1} \times S^q) = \emptyset$. Then $\sigma^*(f)$ is a connected sum of $f: S^{p+q} \rightarrow S^m$ and $s: S^p \times S^q \rightarrow S^m$ (in the sense of Definition of σ^*).

Since $\sigma^*(f) = 0$ it follows that this connected sum extends to an embedding $F: S^p \times D^{q+1} \rightarrow D_{+\epsilon}^{m+1}$. The embedding F is shown in Figure 7 above both dashed lines. Add the trace of the surgery performing the above connected summation to the embedding F . This trace is shown in Figure 7 between the dashed lines. We get an embedding $F': S^p \times D_+^{q+1} - \text{Int } B \rightarrow D_+^{m+1}$, whose restriction to the boundary is the union of $f: S^{p+q} \rightarrow S^m$ and $s: S^p \times S^q \rightarrow S^m$. Add the standard embedding $S^p \times D_-^{q+1} \rightarrow D_-^{m+1}$ to the embedding F' . This embedding is shown in Figure 7 below both dashed lines. We get an embedding $F'': S^p \times S^{q+1} - \text{Int } B \rightarrow S^{m+1}$. Assume that all the above embeddings meet the boundary “regularly”, i.e., their gluing F'' is *smooth*.

It follows by definition that the latter embedding is B -standardized.

Define the embedding $g: D^p \times S^{q+1} \rightarrow S^{m+1}$ to be the restriction of $F'': S^p \times S^{q+1} - \text{Int } B \rightarrow S^{m+1}$. Then by definition $\partial^* g = f$.

The proof of Theorem 1.6 is completed. \square

4 Applications

Finiteness criterion

Now let us apply the sequence of Theorem 1.6 to determine precisely when the set $E^m(S^p \times S^q)$ is finite.

Denote by $E_{\mathbb{U}}^m(S^p \sqcup S^q)$ be the group of isotopy classes of smooth embeddings $S^p \sqcup S^q \rightarrow S^m$, whose restrictions to *both* components S^p and S^q are unknotted. For a finitely generated abelian group G identify $G \otimes \mathbb{Q}$ with $\mathbb{Q}^{\text{rk } G}$. We are going to use tacitly the following isomorphisms (see [10, Theorem 2.4] and [4, Theorem 1.13], respectively):

$$\begin{aligned} E^m(S^p \sqcup S^q) &\cong E_0^m(S^p \sqcup S^q) \oplus E^m(S^q) \cong E_{\mathbb{U}}^m(S^p \sqcup S^q) \oplus E^m(S^p) \oplus E^m(S^q), \\ E^m(D^p \times S^q) \otimes \mathbb{Q} &\cong E^m(S^q) \otimes \mathbb{Q} \oplus \pi_q(V_{m-q,p}) \otimes \mathbb{Q}. \end{aligned}$$

Proof of Corollary 1.7 Let us prove that the map $\partial^*: E^m(D^p \times S^q) \rightarrow E_0^{m-1}(S^{p+q-1} \sqcup S^{q-1})$ has finite image for $m > 2p + q + 2$, $m < p + 3q/2 + 2$. If $p = 0$ then this follows immediately from Theorem 1.6 because the map $i^*: E^m(S^0 \times S^q) \rightarrow E^m(D^0 \times S^q)$ is surjective. Assume further that $p \geq 1$. By the first definition of ∂^* it suffices to prove that at least one of the groups $E^m(D^p \times S^q)$ and $\pi_{q-1}(S^{m-p-q-1})$ is finite. The assumptions $m > 2p + q + 2$ and $m < p + 3q/2 + 2$ imply that $m \leq 2q$. So by Theorem 1.2 the group $E^m(D^p \times S^q)$ is finite unless $4 \mid q + 1$ and by the Serre theorem $\pi_{q-1}(S^{m-p-q-1})$ is finite unless $4 \mid q$. So the map ∂^* has finite image.

This implies that the sequence of Theorem 1.6 tensored by \mathbb{Q} splits for $m < p + 3q/2 + 2$. By the isomorphisms stated in the beginning of this section the corollary follows. \square

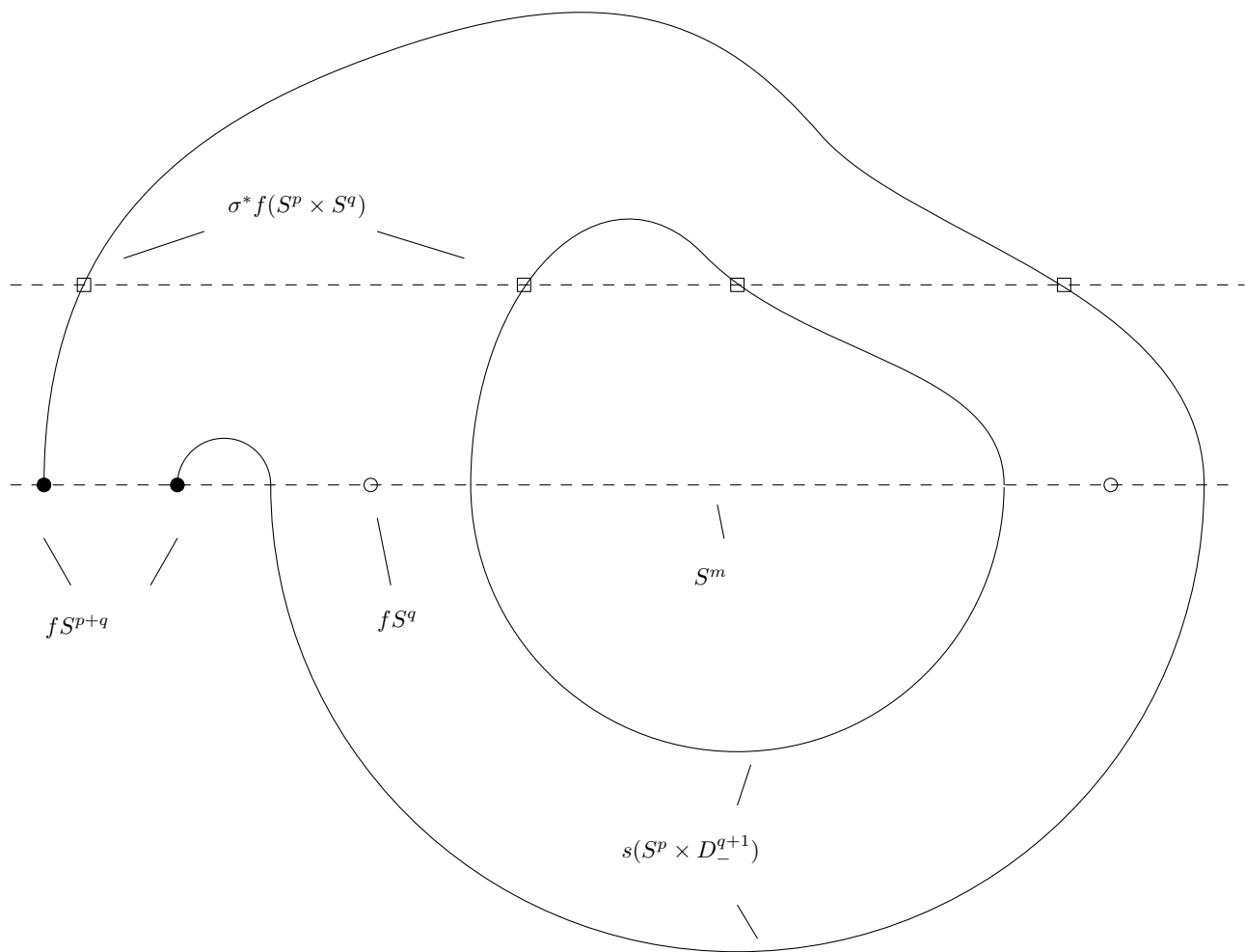


Figure 7: The proof that $\text{Im } \partial^* \supset \text{Ker } \sigma^*$ ($p=0, q=0, m=1$)

Proof of Theorem 1.4 (1) “Only if” part. If neither conditions in the list of Theorem 1.4 hold then all the groups $E^m_{\mathbb{U}}(S^{p+q} \sqcup S^q)$, $E^m(S^{p+q})$, and $E^m(D^p \times S^q)$ are finite by Theorems 1.1–1.3. By Corollary 1.7 (or alternatively by the exactness at the term $E^m(S^p \times S^q)$ in Theorem 1.6) the result follows.

(2) Case when $q + 1$ is divisible by 4. If $m \leq p + 3q/2 + 1/2$ then by Theorem 1.2 the group $E^m(D^p \times S^q)$ is infinite. If $m = p + 3q/2 + 3/2$ then $2(m - p - q - 2) + (m - q - 2) = (m - 3)$ and $m - p - q$ is odd; thus $(2; 1) \in FCS(m - p - q, m - q)$ and by Theorem 1.3 the group $E^m_{\mathbb{U}}(S^p \sqcup S^q)$ is infinite. By Corollary 1.7 the result follows.

(3) The rest of “if” part. If $p + q + 1$ is divisible by 4 then by Theorem 1.1 the group $E^m(S^{p+q})$ is infinite because $m < p + 3q/2 + 2$. If there is $(x, y) \in FCS(m - p - q, m - q)$ such that $(m - p - q)x + (m - q)y = (m - 3)$ then by Theorem 1.3 the group $E^m_{\mathbb{U}}(S^p \sqcup S^q)$ is infinite. By Corollary 1.7 the result follows. \square

Knotted connected sums

Let us give the following easy example of another application of our approach:

Theorem 4.1 For each $q_1 \geq p_1 \geq p_2$, $m > 2p_1 + q_1 + 2$, $q_2 + p_2 = q_1 + p_1$ there is an exact sequence

$$E^m(S^{p_1+q_1}) \rightarrow E^m(S^{p_1} \times S^{q_1}) \oplus E^m(S^{p_2} \times S^{q_2}) \rightarrow E^m(S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2}).$$

Here $E^m(S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2})$ is a set with the marked element — the connected sum of two standard embeddings $S^{p_i} \times S^{q_i} \rightarrow S^m$. This result and Claim 2.4 below imply that if at least one set $E^m(S^{p_i} \times S^{q_i})$ is infinite then the set $E^m(S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2})$ is infinite. Let us remark that for an arbitrary (almost) parallizable $(p - 1)$ -connected manifold N the group $E^m(S^p \times S^q)$ acts on the set $E^m(N)$ [20].

For the proof of Theorem 4.1 we need a definition and a lemma. Let N be a closed connected n -manifold. Denote by $c: S^p \times D^{n-p}_- \rightarrow N$ a fixed embedding. A map $f: N \rightarrow S^m$ is called c -standardized if

- $f \circ c: S^p \times D^{n-p}_- \rightarrow D^m_-$ is the restriction of the standard embedding $S^p \times S^{n-p} \rightarrow S^m$ and
- $f(N - \text{Im } c) \subset \text{Int } D^m_+$.

An isotopy $f_t: N \rightarrow S^m$ is c -standardized, if for each $t \in I$ the embedding f_t is c -standardized.

Lemma 4.2 [20, Standardization Lemma] Assume that $m > n + p + 2$. Then

- (a) any embedding $N \rightarrow S^m$ is isotopic to an c -standardized embedding;
- (b) any isotopy between c -standardized embeddings $N \rightarrow S^m$ is isotopic relatively to the ends to an c -standardized isotopy.

This lemma allows to define an action $\#_c: E^m(S^p \times S^q) \times E^m(N) \rightarrow E^m(N)$ analogously to the group structure on $E^m(S^p \times S^q)$ in § 2, see [20] for details. However, we do not need this action for our proof.

Proof of Theorem 4.1 Denote by $s_i: S^{p_i} \times S^{q_i} \rightarrow S^m$ are the standard embeddings, and by $c_i: S^{p_i} \times D^{q_i}_- \rightarrow S^{p_i} \times S^{q_i} \# S^{p_2} \times S^{q_2}$ the obvious inclusions.

The map $a: E^m(S^{p_1} \times S^{q_1}) \oplus E^m(S^{p_2} \times S^{q_2}) \rightarrow E^m(S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2})$ is the connected summation. The homomorphism $\sigma^*: E^m(S^{p_1+q_1}) \rightarrow E^m(S^{p_1} \times S^{q_1}) \oplus E^m(S^{p_2} \times S^{q_2})$ is defined by the formula $\sigma^*(f) = (s_1 \# f, s_2 \# (-f))$, where $-f = r_m f r_{p_1+q_1}$ is the inverse in the group $E^m(S^{p_1+q_1})$.

To prove exactness we need to show that $a \circ \sigma^*$ is constant and σ^* surjects onto the preimage of the marked element.

For any $f \in E^m(S^{p+q})$ the embedding $a(\sigma^*(f)) = s_1 \# f \# s_2 \# (-f) = s_1 \# s_2$ is the marked element, thus $a \circ \sigma^*$ is constant.

Let us prove that σ^* surjects onto the preimage of $s_1 \# s_2$. Let $\bar{\sigma}_i^*: E^m(S^{p_i} \times S^{q_i}) \rightarrow E^m(S^{p+q})$, where $i = 1, 2$, be the map defined in the proof of Claim 2.4.

Let us define also a map $\bar{a}_1: E^m(S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2}) \rightarrow E^m(S^{p_1} \times S^{q_1})$. Take an embedding $f: S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2} \rightarrow S^m$. By Lemma 4.2.a it is isotopic to a c_2 -standardized embedding $f': S^{p_1} \times S^{q_1} \# S^{p_2} \times S^{q_2} \rightarrow S^m$. Perform the standard embedded surgery on f' along the meridian $f'c_2(S^p \times 0)$. By definition, $\bar{a}_1(f)$ is the class of the obtained embedding $S^{p_1} \times S^{q_1} \rightarrow S^m$.

Take any pair of embeddings $f_i: S^{p_i} \times S^{q_i} \rightarrow S^m$, where $i = 1, 2$. It is easy to see that $\bar{a}_1(f_1 \# f_2) = f_1 \# \bar{\sigma}_2^* f_2$ and $\bar{\sigma}_1^* \bar{a}_1(f_1 \# f_2) = \bar{\sigma}_1^* f_1 \# \bar{\sigma}_2^* f_2$. Now assume that $f_1 \# f_2 = s_1 \# s_2$ is the marked element. Then $\bar{a}_1(f_1 \# f_2) = s_1$ and $\bar{\sigma}_1^* \bar{a}_1(f_1 \# f_2) = 0$. Thus $f_1 \# \bar{\sigma}_2^* f_2 = s_1$, $\bar{\sigma}_1^* f_1 \# \bar{\sigma}_2^* f_2 = 0$. Hence $f_1 = s_1 \# (-\bar{\sigma}_2^* f_2) = s_1 \# (\bar{\sigma}_1^* f_1)$. Analogously $f_2 = s_2 \# (-\bar{\sigma}_1^* f_1)$. So $(f_1, f_2) = \sigma^*(\bar{\sigma}_1^* f_1)$ belongs to the image of σ^* , which proves the theorem. \square

Open problems

There are many similar open questions (by A. Skopenkov):

- (1) Does the set of *piecewise linear* embeddings $S^p \times S^q \rightarrow S^m$ up to piecewise linear isotopy admit a natural group structure for each $m > p + q + 2$? Can the restriction $m > 2p + q + 2$ be weakened to $m > p + q + 2$ in Theorem 1.6 in the piecewise linear category?
- (2) How many embeddings $S^1 \times S^5 \rightarrow S^{10}$ are there up to isotopy? Find more explicit classification results.
- (3) Is it true that for $p \geq 1$ and $m > 2p + q + 2$ there is an isomorphism

$$E^m(S^p \times S^q) \cong E^m(D^{p+1} \times S^q) \oplus E^m(S^{p+q}) \oplus \text{Ker } \lambda,$$

where $\lambda: E^m_U(S^{p+q} \sqcup S^q) \rightarrow \pi_q(S^{m-p-q-1})$ is the linking number?

- (4) When the set of embeddings is finite? Find more finiteness results for the sets $E^m(N)$.

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